# Blending Neural Control Density Functions for Stabilization and Safety


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## Problem of Interest

- **Neural Network based control policies** is popular with its applications in robotics.
- Learning policies is **insufficient** and often requires learning a **certificate**, often a lyapunov function.
- Can we construct a **single** powerful neural control policy using **multiple certifiable neural control policies** without learning a extra certificate?

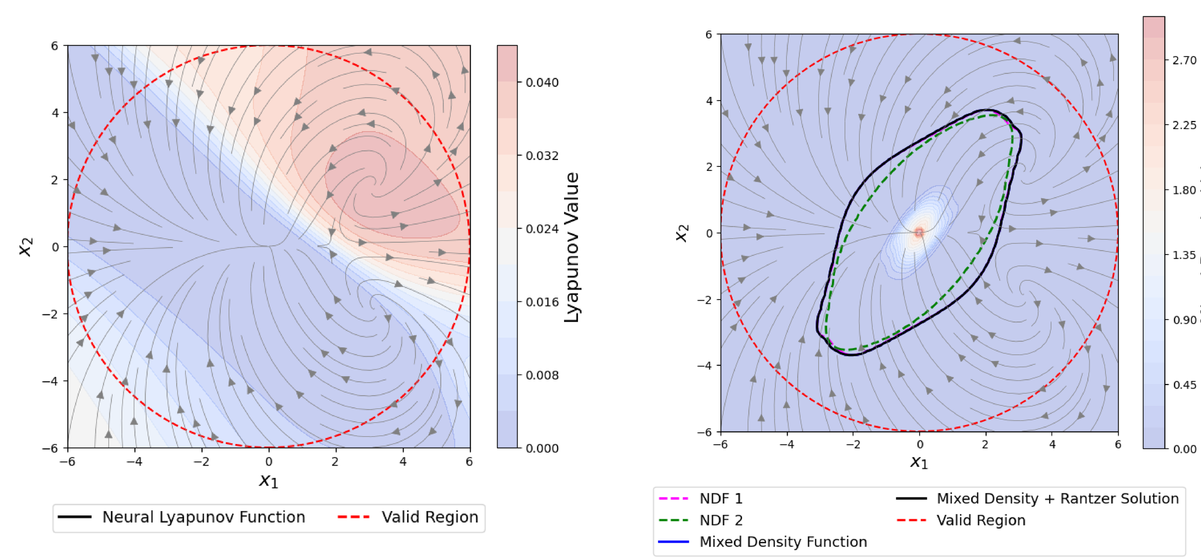
## Challenges

Lyapunov based methods suffer from two critical drawbacks:

- **Multiple equlibiria:** No stability certification when the system has multiple equilibria!
- **No blending:** hard to combine different stabilizing controller while ensuring stability guarantee.

Consider the problem of certification for the given system:

$$\dot{x}_1 = -2x_1 + x_1^2 - x_2^2 \quad \dot{x}_2 = -6x_2 + 2x_1x_2$$




## Density Functions – A Dual Alternative

### Stability via Density Function [1]

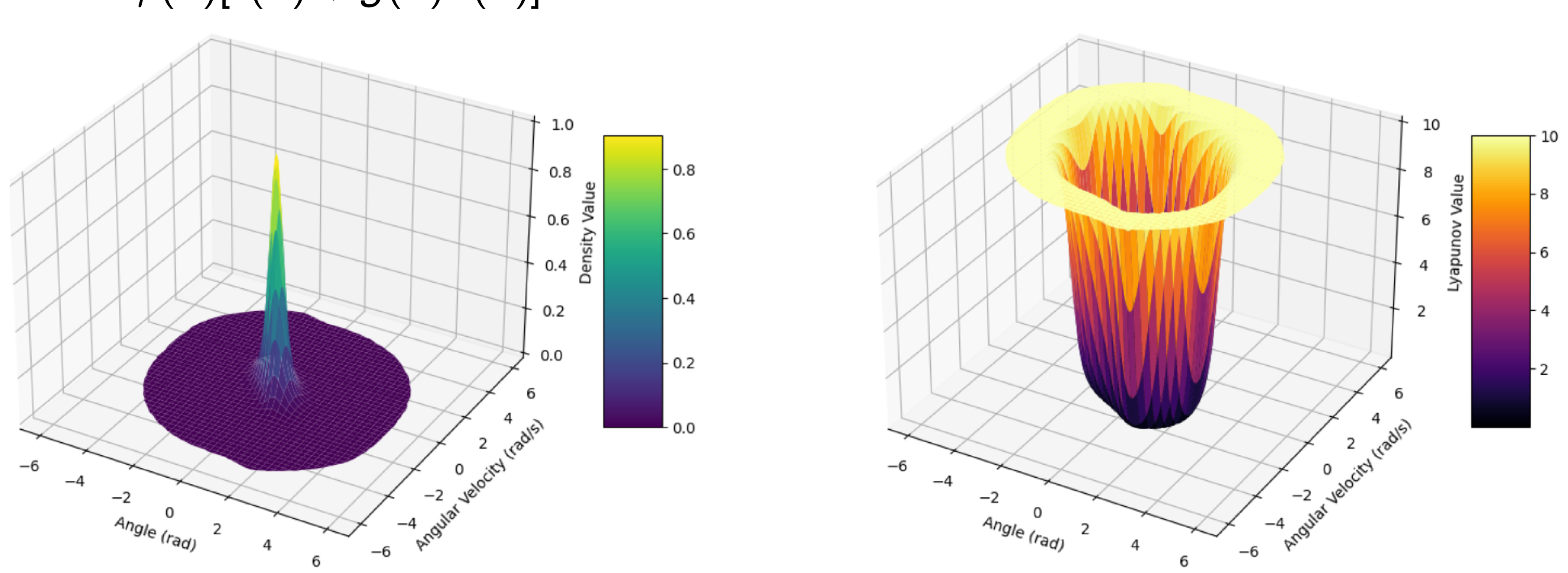
A system is **a.e. stable** if there exists $\rho$ such that:

- $\rho(x) > 0$ for all $x$
- $\int_{x:\|x\|\geq 1} \frac{\rho(x)[f(x)+g(x)u(x)]}{\|x\|} < \infty$
- $\nabla \cdot \rho(x)[f(x) + g(x)u(x)] > 0$

### Stability via Lyapunov Function

A system is **stable** if there exists $V$ such that:

- $V(x) > 0$ for all $x \backslash \{0\}$
- $V(0) = 0$
- $\nabla V(x)^T [f(x) + g(x)u(x)] < 0$




Comparison between Density and Lyapunov Functions certifying stability of Inverted Pendulum.

## Our Contributions

Density function **certifies** the a.e. stability in presence of multiple equilibria and enables blending of multiple controllers.
But parameterizing density function using neural network **does not work**!

- Novel **exponential parameterization** to learn NCDFs certifying a.e. stability.
- Show the **blended controllers** using density **improves RoA** as an advantage of density based certificates.
- Synthesize **safe and stable controllers** via density and barrier functions to address.

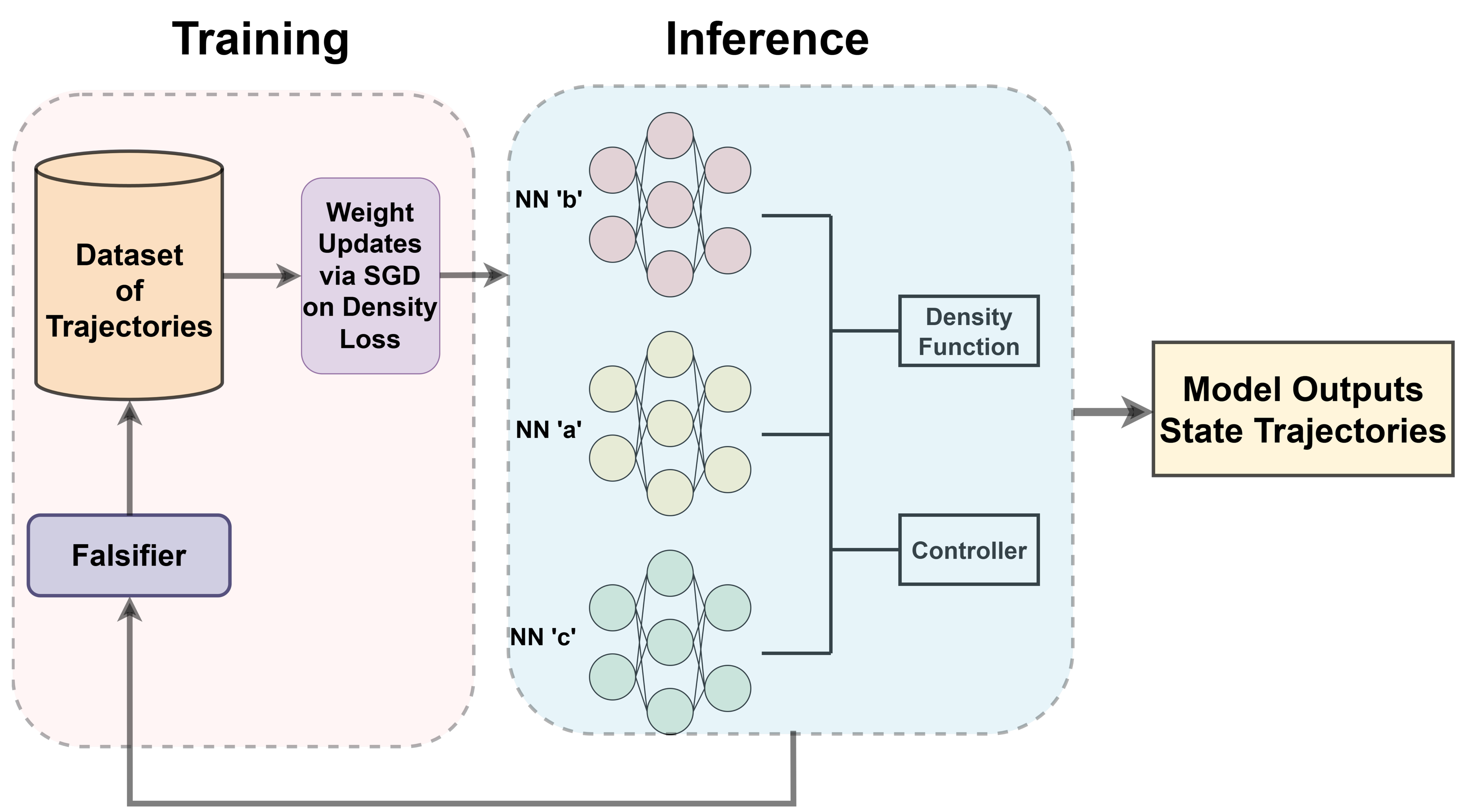
## Neural Control Density Functions (NCDFs)

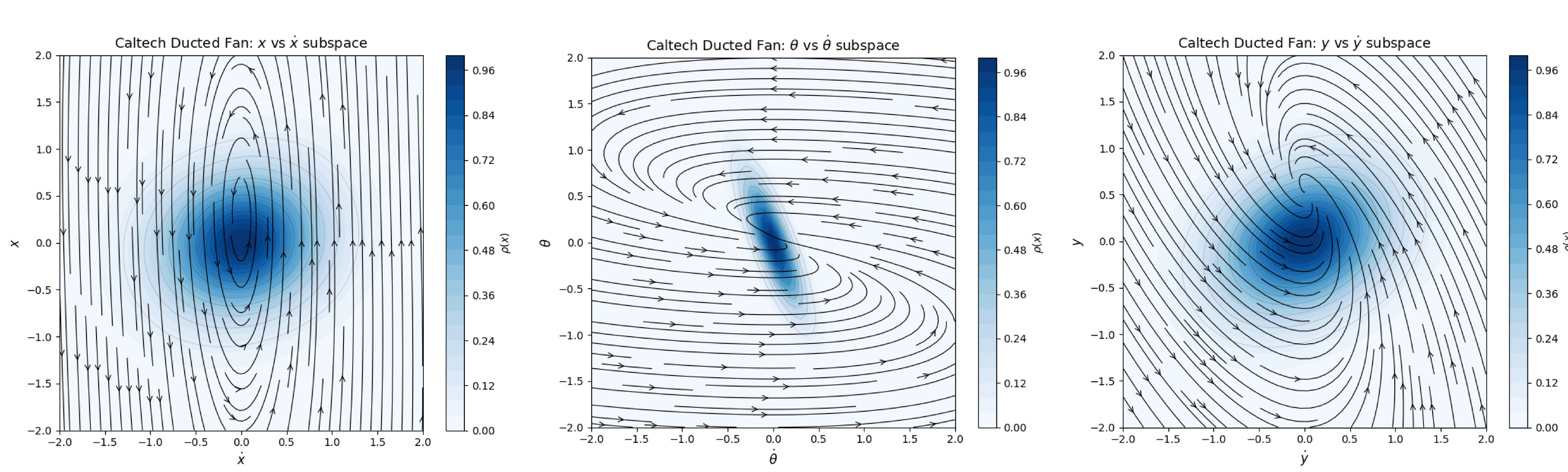
- Set $\psi(x) := \rho(x)u(x)$ and search over $(\psi, \rho)$:

$$\rho(x) = \frac{a(x)}{\exp\left(\|x\|^2 + \|b(x)\|^2\right)} \quad \psi(x) = \frac{c(x)}{\exp\left(\|x\|^2 + \|b(x)\|^2\right)}$$

- **Search over $(\psi, \rho)$ can now be done using $(a, b, c)$–each function parameterized by a Neural Network!**
- **Training Objective:**

$$\mathcal{L}(\theta) = \mathbb{E}\left[\max(0, -\rho_\theta(x)) + \max(0, -\nabla \cdot [\rho_\theta f + g\psi_\theta](x))\right]$$

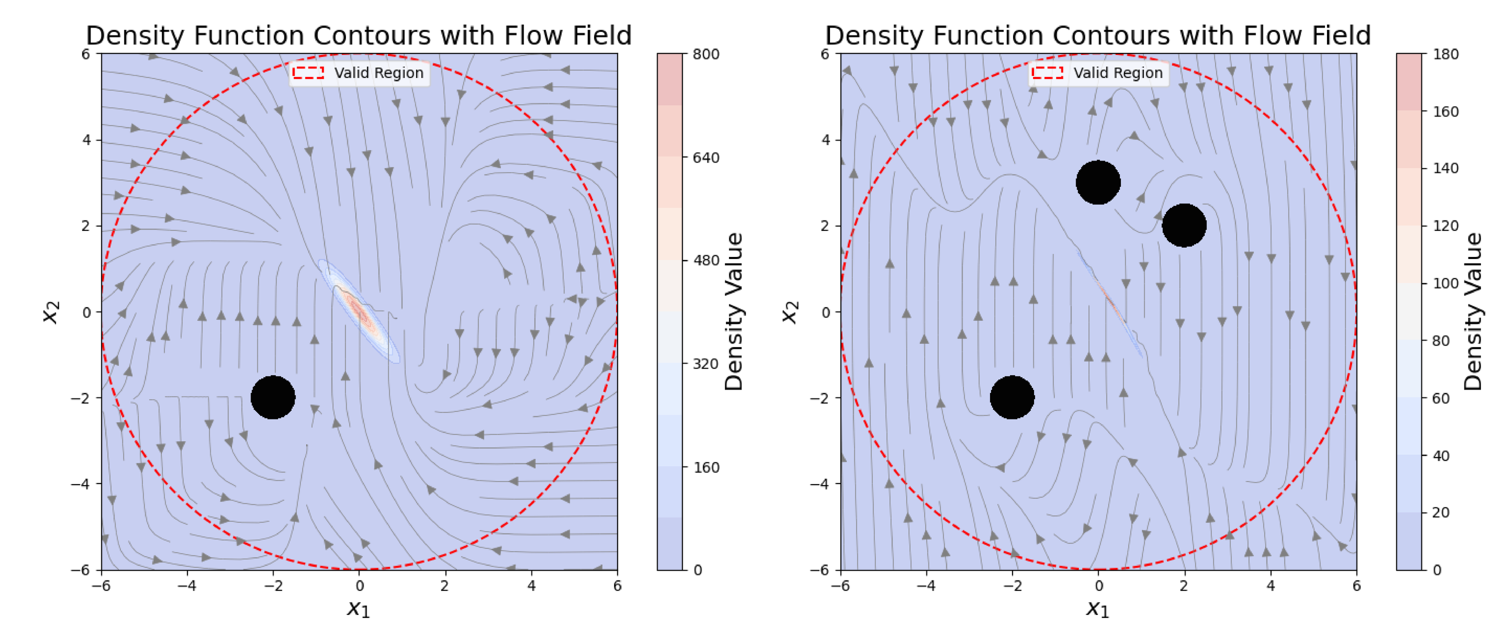



## Safe Stabilization

- Want to remain in **safe set** $\mathcal{C} := \{x : h(x) > 0\}$.
- A control is safe if, $L_f h(x) + L_g h(x)u(x) \geq -\alpha(h(x))$
- **Training Objective:**

$$\mathcal{L}_{\text{safe}}(\theta) = \mathcal{L}(\theta) + \mathbb{E}[\max(0, -[\rho(x)[L_f h(x) + \alpha(h(x))] + \psi(x)L_g h(x)])]$$



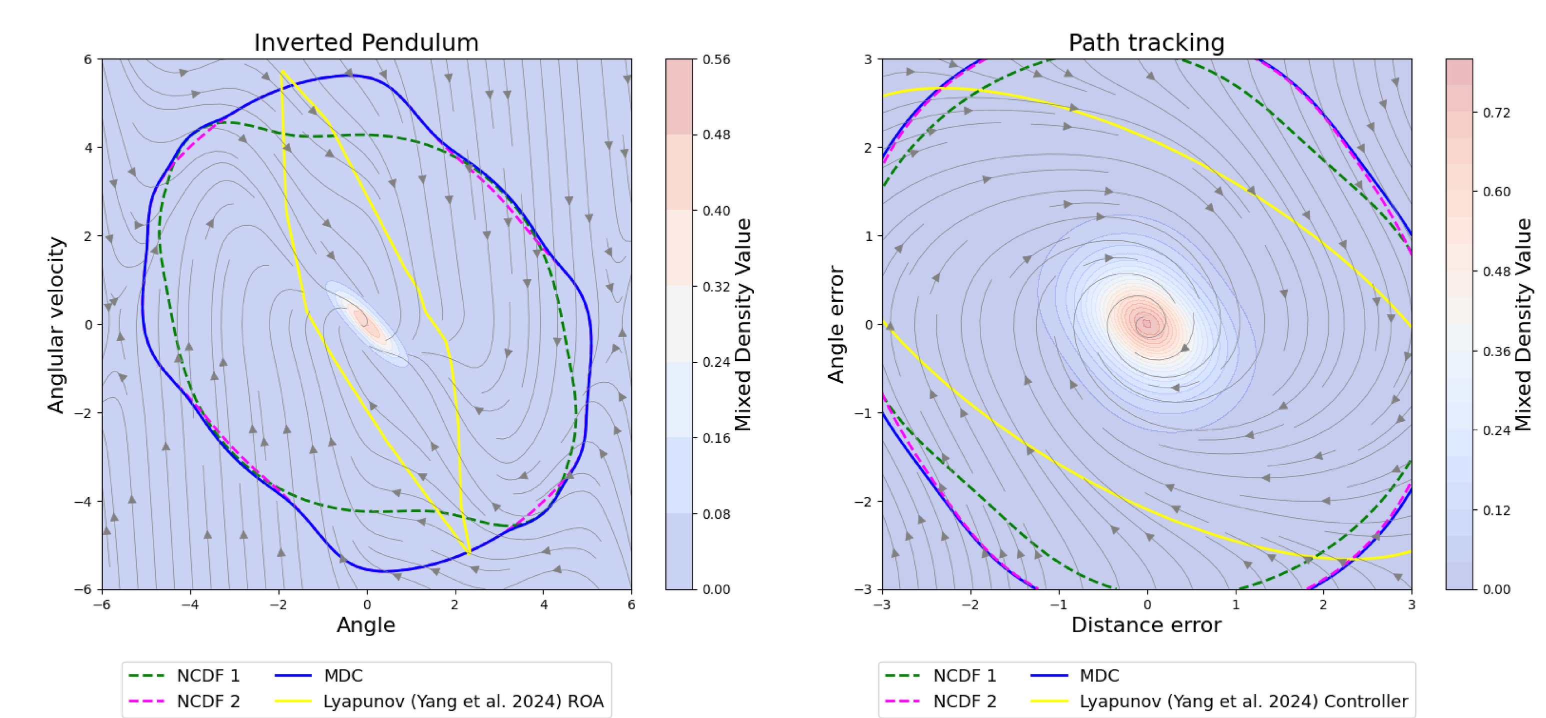

## Expanding Regions of Attraction (RoA)

- Given $N$ different density functions $\rho_i$ + controllers $u_i$, **blending** gives

$$\rho^*(x) = \sum_{i=1}^{N} \rho_i(x), \quad u^*(x) = \sum_{i=1}^{N} \frac{\rho_i(x)u_i(x)}{\rho^*(x)}$$

- **[RoA Expansion Guarantee]** Given: $D_i \Rightarrow$ ROA of $u_i$ for each $i$, $D^\star \Rightarrow$ ROA of $u^\star$. Then,

**Union of $D_i \subseteq D^\star$**




## References


[1] Rantzer, A. A dual to lyapunov's stability theorem. Systems & Control Letters, 42(3):161–168, 2001.

[2] Yang, L., Dai, H., Shi, Z., Hsieh, C.-J., Tedrake, R., and Zhang, H. Lyapunov-stable neural control for state and output feedback: A novel formulation. In International Conference on Machine Learning, pp. 56033–56046. PMLR, 2024.